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Potential flow past a sinusoidal wall of finite amplitude

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Summary

The classical regular perturbation problem of plane potential flow past a sinusoidal wall is pursued via series extension. Fifty terms of the series in non-dimensional wall height ϵ are produced by computer. Analysis reveals convergence to be limited by a branch point at $\epsilon = \pm i$. The series is recast using an Euler transformation and also summed using Padé approximants to yield accurate answers for higher real values of ϵ.

1. Introduction

The problem of plane potential flow past a sinusoidal wall, although familiar as a classical perturbation problem, is one which has received little in-depth treatment in the literature. Kaplan [1] gave this problem its most complete treatment, having chosen it as a model that, regardless of its simplicity, exhibited many of the possible mathematical troubles that occurred in iteration procedures commonly used at that time. I have, after nearly thirty years, chosen the same problem for the same reasons.

Kaplan's approach was two-pronged. He pursued the regular perturbation through several terms by hand calculation, and an integral-equation approach which led to a non-linear equation quite similar to that of Theodorsen and Garrick [2]. His conclusion was that the integral-equation approach was more computationally efficient and therefore superior.

The advent of computers may have changed that. The regular perturbation method now offers an efficient and accurate solution to this problem, and whereas before, radius of convergence and rate of convergence could not be discussed, we now have the tools to analyze these properly.

2. Solution by computer

In terms of the complex velocity potential $F = \phi + i\psi$ and the complex variable $z = x + iy$, the problem of steady, plane, potential flow past a sinusoidal wall, $y = \epsilon \cos x$, is:

Find F, an analytic function of z, where

$$F \to z \quad \text{as} \quad y \to \infty, \quad \text{and} \tag{2.1}$$

$$\mathrm{Im}(F) = \text{constant} \quad \text{on} \quad y = \epsilon \cos x. \tag{2.2}$$

Here (x, y) are non-dimensional rectangular coordinates, ϕ is the non-dimensional velocity potential, and ψ is the non-dimensional stream function.

Equation 2.1 states that the flow tends to a free-stream far from the wall while Eqn. 2.2 states that the wall is a streamline.

The solution can be deduced to be of the form

$$F = z - \mathrm{i}A_0 - \mathrm{i}\sum_{n=1}^{\infty} A_n\,\mathrm{e}^{\mathrm{i}nz} \tag{2.3}$$

thus,

$$\psi = y - A_0 - \sum_{n=1}^{\infty} A_n\,\mathrm{e}^{-ny}\cos nx. \tag{2.4}$$

I assume that the coefficients A_n are functions of ϵ such that

$$A_n \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{2.5}$$

Although Kaplan had faithfully used the method of Stokes [3], solving the problem in the so-called hodograph plane (ϕ, ψ), I did not. There is no physical or mathematical reason to believe that the solution to this problem cannot be pursued in the physical plane, and I found it much more computationally efficient to work there; where, through certain identities involving Bessel functions, the non-linearity arising due to transferring the boundary condition can be reduced to a cubically growing one. In the hodograph plane the non-linearity grows exponentially.

It is possible to pursue some terms by hand calculation. It is found, for example, that

$$\begin{aligned}\psi = y &- \epsilon(\mathrm{e}^{-y}\cos x) - \epsilon^2(1 + \mathrm{e}^{-2y}\cos 2x)/2 - \epsilon^3(\mathrm{e}^{-y}\cos x + 3\,\mathrm{e}^{-3y}\cos 3x)/8\\ &- \epsilon^4(-3 + 5\,\mathrm{e}^{-2y}\cos 2x + 8\,\mathrm{e}^{-4y}\cos 4x)/24\\ &- \epsilon^5(18\,\mathrm{e}^{-y}\cos x + 97\,\mathrm{e}^{-3y}\cos 3x + 125\,\mathrm{e}^{-5y}\cos 5x)/384 + O(\epsilon^6).\end{aligned} \tag{2.6}$$

A pattern may be discerned that continues in the higher-order calculations. That pattern dictates that

$$A_n(\epsilon) = \sum_{j=0}^{\infty} \hat{A}_{n+2j,n}\epsilon^{n+2j}. \tag{2.7}$$

Substituting the above expression into Eqn. 2.4, while making use of the identity

$$\mathrm{e}^{-n\epsilon\cos x} = I_0(n\epsilon) + 2\sum_{k=1}^{\infty}(-1)^k I_{2k}(n\epsilon)\cos kx \tag{2.8}$$

where I_p is the modified Bessel function of the first kind of integer order p, defined by the power series

$$I_p(t) = \sum_{m=0}^{\infty} \frac{(t/2)^{p+2m}}{m!(p+m)!}, \tag{2.9}$$

and requiring that $y = \epsilon \cos x$ to be the $\psi = 0$ streamline, results in this expression:

$$\epsilon \cos x = \sum_{j=1}^{\infty} \hat{A}_{2j,0}\epsilon^{2j}$$

$$+ \sum_{m=1}^{\infty} \epsilon^m \sum_{\substack{n=1,2 \\ n \text{ even if } m \text{ even} \\ n \text{ odd if } m \text{ odd}}}^{m} \cos nx \sum_{j=0}^{(m-n)/2} \frac{\hat{A}_{n+2j,n}(n/2)^{m-n-2j}}{(m-n-2j)!(m-n-2j)!}$$

$$+ \sum_{m=2}^{\infty} \epsilon^m \sum_{n=1}^{m-1} \sum_{\substack{k=1,2 \\ k \text{ chosen such that} \\ m+n+k \text{ is even}}}^{m-n} (-1)^k (\cos(n+k)x$$

$$+\cos(n-k)x) \sum_{j=0}^{(m-n-k)/2} \frac{\hat{A}_{n+2j,n}(n/2)^{m-n-2j}}{((m-n-k-2j)/2)!((m-n+k-2j)/2)!} \tag{2.10}$$

This last expression has summations nested four deep, thus the computational work is growing cubically with the order of ϵ. Not many terms may be calculated by hand before the required effort becomes prohibitive.

This tedious algebra was turned over to an IBM 3033 via a Fortran program consisting of approximately one hundred lines. In about twenty-five seconds fifty terms of this series were produced with coefficients in quadruple precision (32 significant figures). This program was actually written in two totally different ways. Both were run in single, double, and quadruple precision. From comparisons of these it is estimated that the coefficients of the fiftieth term have at least ten accurate significant figures (of the thiry-two computed).

3. Analysis of coefficients

The particular physical quantity that is analyzed is important from one point of view and not from another. Since this problem is truly elliptic and it is a small-disturbance perturbation (no stagnation points in the flow-field) the radius of convergence of any physical quantity near the wall should be independent of the field point (x, y) where it is evaluated, i.e., when the solution breaks down it does so everywhere at the same value of ϵ. This would not be the case were the problem hyperbolic or mixed or even an elliptic problem with stagnation points (as in thin-airfoil theory). In this view the choice of physical quantity is unimportant.

Table 1. Coefficients and ratios of alternating coefficients for series for v_{max} in both non-dimensional wall height ϵ (a_n) and the Euler transform variable β (δ_n)

n	a_n	a_n/a_{n-2}	δ_n	δ_n/δ_{n-2}
1	1.000000000		1.0000000000	
2	0.000000000	0.000000	0.0000000000	0.000000
3	−0.250000000	−0.250000	0.2500000000	0.250000
4	0.0833333333	∞	0.0833333333	∞
5	0.0989583333	−0.395833	0.0989583333	0.395833
6	−0.0656250000	−0.787500	0.1010416666	1.21250
7	−0.0400173611	−0.404386	0.0511284722	0.516667
8	0.0430375397	−0.655706	0.9615575397 − 1	0.951644
9	0.0158339940	−0.395428	0.3526843843 − 1	0.689800
10	−0.0270352587	−0.628277	0.8467109048 − 1	0.880562
11	0.5783014314 − 2	−0.365458	0.3056167737 − 1	0.866545
12	0.1695994837 − 1	−0.627327	0.7250786108 − 1	0.856347
13	0.1653714904 − 2	−0.285961	0.2969087164 − 1	0.971507
14	−0.1080545798 − 1	−0.637116	0.6166543069 − 1	0.850465
15	−0.2626703767 − 4	−0.015884	0.2993352586 − 1	1.00817
16	0.7052585756 − 3	−0.652687	0.5623962430 − 1	0.853379
17	−0.5451703739 − 3	20.7549	0.3029073638 − 1	1.01193
18	−0.4738886795 − 2	−0.671936	0.4530686255 − 1	0.860955
19	0.6825188786 − 3	−1.25194	0.3043397650 − 1	1.00473
20	0.3286522200 − 2	−0.693522	0.3946538640 − 1	0.871069
21	−0.6534041137 − 3	−0.957342	0.3029384660 − 1	0.995396
22	−0.2354009440 − 2	−0.716261	0.3482688811 − 1	0.882467
23	0.5709340207 − 3	−0.873784	0.2989510801 − 1	0.986838
24	0.1739696489 − 2	−0.739035	0.3114654438 − 1	0.894325
25	−0.4807121750 − 3	−0.841976	0.2929052810 − 1	0.979777
26	−0.1323698264 − 2	−0.760879	0.2822061600 − 1	0.906059
27	0.3992370752 − 3	−0.830511	0.2853523772 − 1	0.974214
28	0.1033925726 − 2	−0.781089	0.2588533976 − 1	0.917249
29	−0.3308765960 − 3	−0.828772	0.2767775170 − 1	0.969950
30	−0.8263849697 − 3	−0.799269	0.2401127355 − 1	0.927601
31	0.2753080756 − 3	−0.832057	0.2675771771 − 1	0.966759
32	0.6737533724 − 3	−0.815302	0.2249692433 − 1	0.936932
33	−0.2306865776 − 3	−0.837921	0.2580623443 − 1	0.964441
34	−0.5587239798 − 3	−0.829271	0.2126300493 − 1	0.945152
35	0.1949266239 − 3	−0.844985	0.2484701488 − 1	0.962830
36	0.4700968084 − 3	−0.841376	0.2024766882 − 1	0.952225
37	−0.1661630243 − 3	−0.852439	0.2389770321 − 1	0.961794
38	−0.4004590399 − 3	−0.851865	0.1940270002 − 1	0.958268
39	0.1428700734 − 3	−0.859819	0.2297108826 − 1	0.961226
40	0.3447905985 − 3	−0.860988	0.1869051829 − 1	0.963295
41	−0.1238498419 − 3	−0.866870	0.2207613761 − 1	0.961041
42	−0.2996136076 − 3	−0.868973	0.1808184252 − 1	0.967434
43	0.1081793053 − 3	−0.873471	0.2121884688 − 1	0.961167
44	0.2624650548 − 3	−0.876012	0.1755387195 − 1	0.970801
45	−0.9515220287 − 4	−0.879579	0.2040292403 − 1	0.961547
46	−0.2315640051 − 3	−0.882266	0.1708887019 − 1	0.973510
47	0.8422817580 − 4	−0.885194	0.1963033476 − 1	0.962133
48	0.2055976165 − 3	−0.887865	0.1667306132 − 1	0.975668

This is an appropriate place to mention the work of Bollmann [4], who has recently published a solution via series extension to the problem of transonic flow past a sinusoidal wall of finite height. This problem incorporates the mathematical complexities of being of mixed type. Its behavior is thus much different from the problem at hand.

On the other hand the rate of convergence and overall behavior of the series of all physical quantities are not the same. Some physical quantities may be very poorly convergent and have power series that are ill-behaved. Therefore, from a series-analysis viewpoint, the choice of quantity does matter.

The important physical quantities in this problem really only number two; the velocity (u, v) or equivalently the speed $Q = (u^2 + v^2)^{1/2}$, and the pressure coefficient C_p (simply $1 - Q^2$).

For the velocity components, the slowest converging quantities are the maximum and minimum velocities on the wall. The maximum wall velocity occurs at the field point $(0, \epsilon)$ while the minimum occurs at $(\pi, -\epsilon)$. These quantities are then poor candidates for analysis to determine radius of convergence. I prudently choose another velocity component to analyze, one with exceptionally good behavior. It is the x-component of the velocity on the wall at $(\pi/2, 0)$. The total speed may be a better choice but recall that at this point the y-velocity to x-velocity ratio is simply $-\epsilon$.

The coefficients for the maximum fluid velocity on the wall v_{max} and x-velocity of the fluid on the wall where the wall crosses the $y = 0$ plane v_{mid} appear in Tables 1 and 2

Table 2. Even coefficients and ratios of successive even coefficients (odd coefficients are zero) for series v_{mid} in nondimensional wall height $\epsilon(a_n)$ and the Euler transform variable $\beta(\delta_n)$

n	a_n	a_n/a_{n-2}	δ_n	δ_n/δ_{n-2}
2	−1.00000000000	−1.000000	−1.000000000000	−1.00000
4	0.91666666667	−0.916667	−0.083333333333	0.083333
6	−0.84687500000	−0.923864	−0.013416666667	0.161000
8	0.79342757936	−0.936889	0.003177579365	−0.236838
10	−0.75204856375	−0.947848	0.007828420380	2.46364
12	0.71914378685	−0.956246	0.009010095099	1.15095
14	−0.69224089948	−0.962590	0.009194952714	1.02052
16	0.66969534154	−0.967431	0.009210782308	1.00172
18	−0.65040579746	−0.971196	0.009342128426	1.01426
20	0.63361865662	−0.974190	0.009677996022	1.03595
22	−0.61880414369	−0.976619	0.01023572012	1.05763
24	0.60557933572	−0.978628	0.01100785258	1.07543
26	−0.59366005730	−0.980317	0.01198017081	1.08833
28	0.58283030692	−0.981758	0.01313833948	1.09667
30	−0.57202238449	−0.983000	0.01447005543	1.10136
32	0.563803655235	−0.984084	0.01596543796	1.10334
34	−0.55536749402	−0.985037	0.01761679568	1.10343
36	0.54752699028	−0.985992	0.01941822565	1.10226
38	−0.54021039658	−0.986637	0.02136521776	1.10027
40	0.53335784295	−0.987315	0.02345432022	1.09778
42	−0.52691887984	−0.987927	0.02568287523	1.09502
44	0.52085062459	−0.988483	0.02804881723	1.09212
46	−0.51511633981	−0.988990	0.03055052199	1.08919
48	0.50968433348	−0.989454	0.03318669565	1.08629
50	−0.50452707409	−0.989881	0.0359563096	1.08345

respectively. Note that the coefficients for the minimum velocity on the wall are those of the maximum with the odd-numbered signs reversed.

Examination of Table 1 reveals an asymptotic pattern of signs $(++--)$ which indicates a complex conjugate pair of singularities on the imaginary axis of ϵ. Such a sign pattern is modelled by the simple function $(1+b\epsilon+c\epsilon^2)/(1+\epsilon^2)=1+b\epsilon-(1-c)\epsilon^2-b\epsilon^3+(1-c)\epsilon^4+b\epsilon^5+\ldots$ My solution should not be stopped short of the wall of infinite amplitude by any physics we have neglected. It is common to encounter mathematical singularities on the negative real axis of some physical parameter (low-Reynolds-number expansions, for instance). But here the only acceptable singularities are mathematical ones lying off the real axis, for the negative real axis simply represents a phase shift of the wall. And the singularities must be in complex conjugate pairs to make the coefficients real.

I proceed using methods popularized by Gaunt and Guttmann [5], Van Dyke [6] and others. Consider a function:

$$f(\epsilon)=\sum_{n=0}^{\infty} a_n\epsilon^n \tag{3.1}$$

where f has a complex conjugate pair of singularities on the imaginary axis and is of the form

$$f(\epsilon)=K(1+b\epsilon+c\epsilon^2)\left(\epsilon_0^2+\epsilon^2\right)^{\alpha}\quad (\text{singularity at} \pm \mathrm{i}\epsilon_0). \tag{3.2}$$

Then the behavior of the ratio R of successive odd or successive even coefficients as $n\to\infty$ is

$$\frac{a_{2n}}{a_{2n-2}}\sim\frac{a_{2n+1}}{a_{2n-1}}\sim\frac{-1}{\epsilon_0^2}\left(1-\frac{\alpha+1}{n}\right). \tag{3.3}$$

I hypothesize that this result would be true regardless of the nature of the part of $f(\epsilon)$ which is analytic at $\epsilon=\pm\mathrm{i}\epsilon_0$.

The ratio of every other coefficient was computed and also appears in Tables 1 and 2. It can be seen that these ratios are, at least asymptotically, fairly well-behaved. I have plotted in Fig. 1 the absolute value of these ratios for $v_{\max}$ versus $1/n$ as suggested by Domb and Sykes [7], thereby having a graphical representation of what the radius of convergence might be. It suggests that $\epsilon_0=1.00$.

Table 2 shows clearly that the series for v_{mid} is much better behaved. The pattern of signs establishes itself immediately $(+0-0)$. This, too, is an indication of a complex conjugate pair of singularities on the imaginary axis of ϵ. The model function used before easily produces this result by setting b equal to zero. The ratio a_n/a_{n-2} is seen to monotonically approach -1.00 after the first term. The Domb-Sykes plot, Fig. 2, of these ratios also strongly suggests a radius of convergence of 1.00. I can refine this guess of the radius of convergence by fitting successive polynomials in $1/n$ to the ratios. This is easily done through the formation of a Neville table (Gaunt and Guttmann [5]). The lower left-hand portion of this table appears in Table 3. It confirms that the radius of convergence is 1.00 to within three decimals.

It is also possible from considerations of the Domb-Sykes plots, Figs. 1 and 2, to estimate the nature of the nearest singularity. For $v_{\max}$ the Domb-Sykes plot suggests that

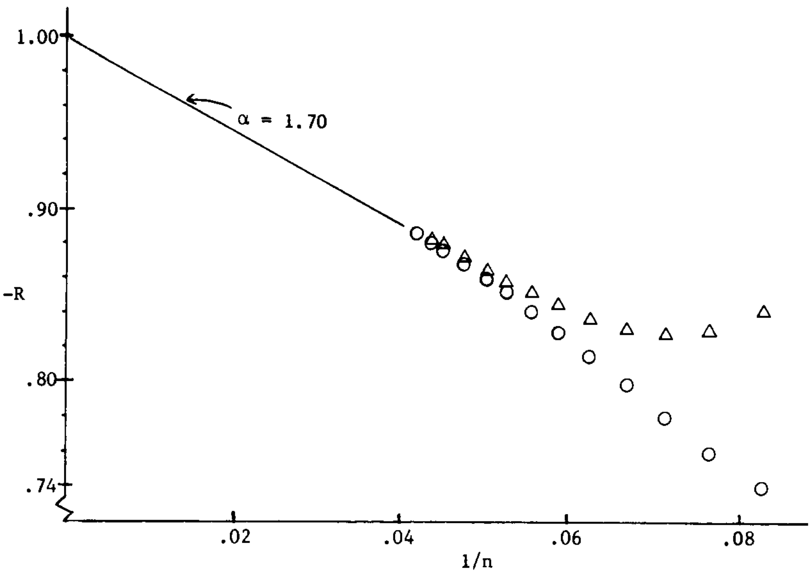

Figure 1. Graphical ratio test of Domb and Sykes applied to successive even (○) and successive odd (△) coefficients of series for v_{max}.

the exponent α is approximately 1.70, while Fig. 2 suggests that α for v_{mid} is approximately -0.75. That these two are different is somewhat surprising. One might expect that the nature of the nearest singularity does not change with the field point just as the radius of convergence does not change. But I can cite no reason why this must be so and after much thought have abandoned my stand that it must be.

Another estimate of the radius of convergence can be gained by the formation of near-diagonal Padé approximants (Baker [8]). These approximants perform analytic continuation by picking out any poles of the function and by placing a line of poles along

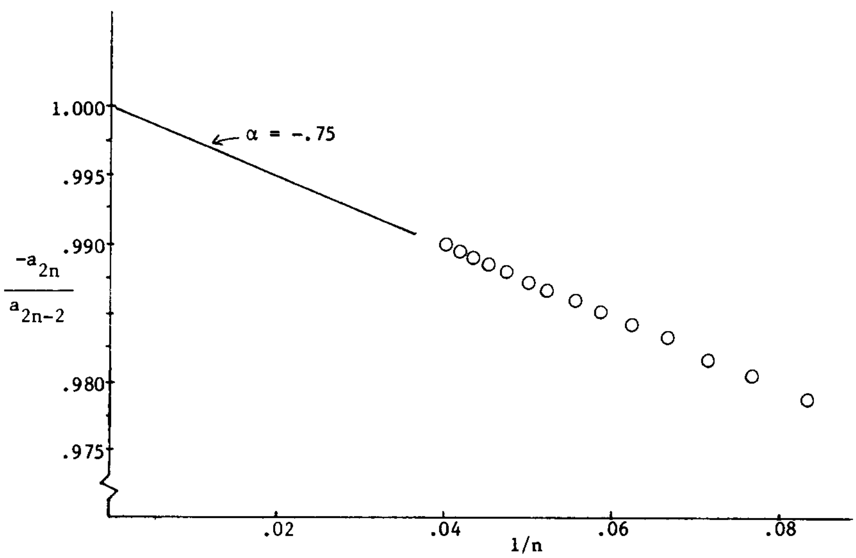

Figure 2. Graphical ratio test of Domb and Sykes for series for v_{mid}. (odd coefficients are zero).

what would be a branch cut. Examination of poles of the three near-diagonal approximants for v_{max} reveals a branch cut emanating from $\epsilon = \pm i$ and extending away from $\epsilon = 0$ slightly to the negative real side of the imaginary axis. A similar procedure for v_{mid} also reveals a branch cut. In the ϵ^2 plane, it emanates from -1 and extends outward along the negative real axis. Thus in both cases branch cuts are suggested and there seems to be no other singularities in the nearby complex plane.

I take the radius of convergence to be 1.00 for all fluid velocities on the wall. This is somewhat higher than what Kaplan had suspected which was approximately 0.7.

Two methods of accelerating convergence and extending the usefulness of series were used. First, the Padé approximants can be used to sum the series. Insofar as the three computed near-diagonal Padé approximants agreed to the fourth decimal place they were taken as accurate. Values of the fluid wall speed at thirty stations between $x = 0$ and $x = \pi$ were computed in this way. (The flow is symmetric about $x = \pi$.) The wall pressure coefficient was then calculated from this result. These two quantities for $\epsilon = 0.25$, 0.50, 0.75, 1.00, and 1.25 appear in Figs. 3 and 4.

Second, an Euler transformation of ϵ^2 was used to extend and improve the two series for fluid velocities. The transformation is

$$\beta^2 = \frac{\epsilon^2}{1+\epsilon^2}, \tag{3.4}$$

so that

$$\beta = \frac{\epsilon}{\sqrt{1+\epsilon^2}} \tag{3.5}$$

and

$$\epsilon = \frac{\beta}{\sqrt{1-\beta^2}}. \tag{3.6}$$

The series become

$$F = \sum_{n=0}^{\infty} \delta_n \beta^n. \tag{3.7}$$

Table 3. Bottom left-hand corner of Neville table for reciprocal square of the radius of convergence $1/\epsilon_0^2$ for v_{mid}

n	Linear fit	Quadratic	Cubic	Quartic	Quintic
40	1.0001972	0.9999772	1.0000325	0.9999914	0.9999611
42	1.0001769	0.9999839	1.0000241	0.9999886	0.9999795
44	1.0001598	0.9999885	1.0000177	0.9999892	0.9999912
46	1.0001452	0.9999917	1.0000131	0.9999911	0.9999979
48	1.0001326	0.9999945	1.0000138	1.0000173	1.0001168
50	1.0001215	0.9999940	0.9999900	0.9998650	0.9992557

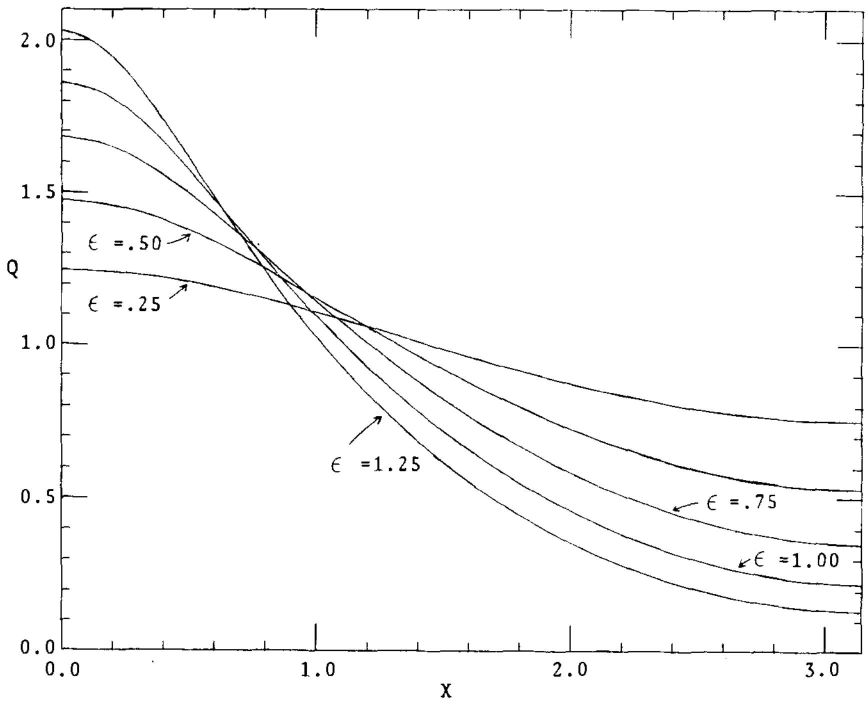

Figure 3. Fluid speed on the wall by Padé approximants.

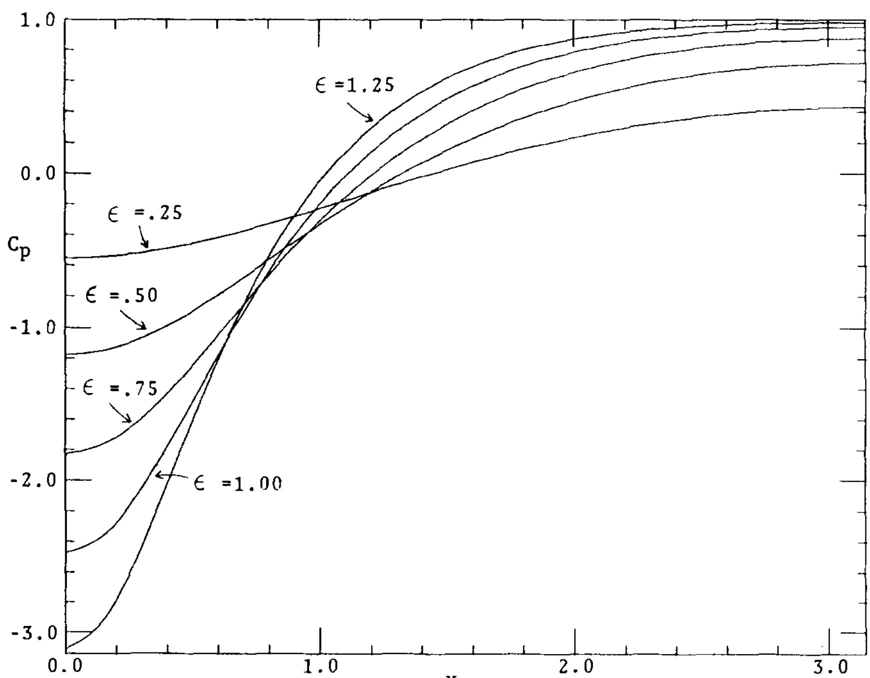

Figure 4. Wall pressure coefficient by Padé approximants.

Table 4. Last eight partial sums for v_{max} and v_{mid} with $\epsilon = 1.25$

n	v_{max}	n	v_{mid}
41	2.028702677	36	0.3585493750
42	2.028703234	38	0.3585511438
43	2.028703744	40	0.3585523278
44	2.028704073	42	0.3585531128
45	2.028704372	44	0.3585536447
46	2.028704568	46	0.3585539943
47	2.028704743	48	0.3585542259
48	2.028704859	50	0.3685543790

The coefficients δ_n and the ratios of alternating coefficients for v_{max} and v_{mid} appear in Tables 1 and 2 respectively. The ratios now suggest a radius of convergence of 1.00. Thus the Euler transformation has successfully extended the usefulness of both series to the whole real axis of ϵ.

Using these transformed series v_{max} and v_{mid} can be accurately calculated for ϵ up to 1.25 and higher. Table 4 shows the last eight partial sums for v_{max} and v_{mid} with $\epsilon = 1.25$. The convergence is monotonic and rapid. It can be seen that at a value of $\epsilon = 1.25$, the maximum wall speed is about 2.028704 times the free-stream speed.

Table 5 shows v_{max} and v_{mid} for various values of ϵ up to 1.25 as calculated by both the Euler transformation and by Padé approximants. The agreement is excellent for all ϵ which gives me confidence in all calculations.

It can be seen in Fig. 3 that with $\epsilon = 1.25$ the maximum speed is twice the free-stream speed and the minimum speed is less than fifteen percent of it. The corresponding graph of C_p in Fig. 4 shows a very large gradient in pressure between $x = 0$ and π. This, of course, is an adverse gradient and any fluid viscosity would undoubtedly cause the flow to separate in this region.

4. Discussion

For this problem of plane potential flow there cannot be a better way to solve it than by the extension and analysis of the small-disturbance perturbation. The method is computationally efficient, especially when pursued in the physical plane. The series can be analyzed and their usefulness extended to walls of finite height by methods of analytic continuation.

Table 5. v_{max} and v_{mid} as computed by Padé sums and by Euler transformed series

ϵ	v_{max}		v_{mid}	
	Padé sum	Last partial sum	Padé sum	Last partial sum
1.25	2.028705281	2.028704859	0.3585546709	0.3585543790
1.00	1.864499528	1.864499524	0.4782200490	0.4672200477
0.75	1.681522381	1.681522381	0.6287056789	0.6287056789
0.50	1.475886296	1.475886295	0.7965676465	0.7965676465
0.25	1.246498136	1.246498135	0.9408854330	0.9408854334

The success of this work suggests new work in some areas. It would be worthwhile to pursue this question of the nature of the nearest singularity, whether it does change with location on the wall. To conclude that it does change will be an important result for this developing technique of series extension. It would also be of value to know something of the solution in the vicinity of the wall apex as the wall height becomes very large. Such a solution would be useful in a real fluid situation on the side with the favorable pressure gradient. These two questions may be pursued through the calculation of additional terms in the series.

Another related problem that could be pursued through small-disturbance perturbation is that of three-dimensional potential flow over a doubly-periodic wall. This problem offers many facets because of the opportunity to change the orientation of the free stream as well as the ratio of wall wavelengths. It is of slight additional difficulty over the plane case when pursued via series extension in the physical plane. But it cannot be pursued in a hodograph plane and an integral-equation approach should prove to be cumbersome.

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